



# MODELING STUDENTS' MATHEMATICS USING STEFFE'S FRACTION SCHEMES

By Anderson H. Norton and Andrea V. McCloskey



*Anderson H. (Andy) Norton, norton3@vt.edu, teaches in the mathematics department at Virginia Tech in Blacksburg, where he also conducts research on children's mathematical development. He concentrates on building models of students' mathematics and studying how students form mathematical conjectures. Andrea V. McCloskey, amcclosk*

*@indiana.edu, is a doctoral student in mathematics education at Indiana University in Bloomington, Illinois. She is interested in supporting the efforts of mathematics teachers, especially their attempts to obtain valid and helpful assessment information.*

Each year, more teachers learn about the successful intervention program known as Math Recovery (USMRC 2008; Wright 2003). The program uses Steffe's whole-number schemes to model, understand, and support children's development of whole-number reasoning. Readers are probably less familiar with Steffe's fraction schemes, which have proven similarly useful in supporting children's development of fractional reasoning. The purpose of this article is to introduce some of these schemes. We provide examples of student work accompanied by discussions of how fraction schemes can be used as tools for insight into student reasoning. We hope that teachers will find the schemes useful in understanding their students as mathematicians.

In their Fractions project, Steffe (2002) and Olive (1999) conducted teaching experiments with six pairs of students over a three-year period—from their third-grade year through their fifth-grade year. Steffe and Olive hypothesized that the students' fractional schemes could be constructed through reorganizations of their whole-number schemes, which Steffe and colleagues identified in previous research (e.g., Steffe, Cobb, and Glasersfeld 1988). As a former student of Steffe, Bob Wright built on this research of students' whole-number schemes by engaging teachers in teaching experiments with students in the early grades (2000; 2003). Wright's increasingly popular Math Recovery program uses teaching experiments as interventions for struggling students, during which teachers challenge students at the "cutting edge" of their knowledge and strategies (2000, p. 142).

We have extended the work of Steffe, Olive, and Wright by engaging elementary school teachers in conducting their own teaching experiments with pairs of students. The teachers posed fraction tasks to the students and observed their problem-solving strategies in an effort to understand how students think. As part of the project, we administered pretests and posttests to all of the students in the participating teachers' classrooms. The test tasks were similar to tasks the teachers posed in the teaching experiment. We use student work from the pretests to provide examples of responses that indicate the fraction schemes described here.

## Operations and Schemes

Schemes are teacher constructs used to model students' cognitive structures. They consist of three components: a template for recognizing situations

in which the scheme applies, mental actions (operations) that are activated when such a situation is recognized, and expected results of operating (see **fig. 1**). Operations constitute the key component of a scheme because schemes are, in essence, ways of operating. For example, a student might be asked to add 8 plus 5, activating the mental action of counting on from 8, which might be represented by the student saying, “8; 9 is one; 10 is two; 11 is three; 12 is four; 13 is five; 13.” Teachers might attribute to this student a scheme for counting on. Situations perceived as involving addition, or joining objects, fit the recognition template and trigger the scheme. This activates the operations of iterating a unit of one and double counting, represented in the student’s verbalization, “9 is one, 10 is two ...” Finally, the student’s actions fit the expected result of the scheme by reaching the addend, “five.” A student might make a mistake in her counting sequence, reaching, for instance, “14 is five,” and her actions could still fit the expected result of reaching “five.”

Schemes differ from strategies in several ways:

- Schemes describe ways of operating that usually occur outside of the student’s awareness.
- Schemes are activated at once, rather than in a procedural manner.
- Schemes are teacher constructs; teachers attribute schemes to students in order to explain students’ actions (including verbalizations).

A scheme fits a teacher’s observations of a student in the same way that a scientific theory fits observations of natural phenomena. The teacher cannot say that the scheme actually exists within the student’s mind any more than Kepler could say that his laws of planetary motion existed within the planets. The teacher attributes the scheme to the student only

**Figure 1**

**Components of an operational scheme**



because it is useful in explaining and predicting the student’s actions.

As mentioned before, operations are mental actions, abstracted from experience to become available for use in various situations. Rotation is an operation that you might have abstracted from experiences of turning your head, spinning your body, or twisting a screwdriver. People use such an operation whenever they recognize a common shape that is turned at an uncommon angle, as often occurs when studying geometry. (Young students often fail to recognize a square when it is not oriented as they expect; such students are not using a rotation operation.) Likewise, we can identify several key operations that students use to understand fractions (Olive 1999; Steffe 2002). We outline these in the next section.

**Fraction operations**

Here we describe several important fraction operations, which we summarize in **Table 1**:

- *Unitizing* produces a unit, or whole, for a student. Anything can be taken as a unit by establishing it as a separate entity, such as extracting foreground from background. Students may also unite a collection of objects into a whole, producing a *composite unit*.
- *Partitioning* a continuous unit (e.g., a candy bar), a continuous composite unit (e.g., three-fifths of

**Table 1**

**Fractional Operations**

Operation	Description	Example of using the operation
<i>Unitizing</i>	Treating an object or collection of objects as a unit, or a whole	Treating two trapezoids as one whole (as with pattern blocks)
<i>Partitioning</i>	Separating the unit, or the whole, into equal parts	Equally sharing a pizza among four people
<i>Disembedding</i>	Imaginatively pulling out a fraction from the whole, while keeping the whole intact	After the pizza has been sliced in fourths, imagining what three-fourths of the pizza would look like
<i>Iterating</i>	Repeating a part to produce identical copies of it	Using a one-fifth piece to identify a three-fifths piece (as with fraction rods)

**Table 2**

**Fractional Schemes**

Scheme	Operations	Example task
<i>Simultaneous partitioning scheme</i>	<i>Unitizing</i> the whole, <i>partitioning</i> the continuous whole using a composite unit as a template	Share this candy bar equally among you and two friends (also see <b>fig. 2</b> ).
<i>Part-whole scheme</i>	<i>Unitizing</i> , <i>partitioning</i> , and then <i>disembedding</i> a part from the partitioned whole	Show me two-thirds of the candy bar (also see <b>fig. 3</b> ).
<i>Equi-partitioning scheme</i>	<i>Unitizing</i> , <i>partitioning</i> , and then mentally <i>iterating</i> any part to determine its identity with the other parts	If you share this candy bar equally among you and two friends, draw what your piece would look like.
<i>Partitive unit fractional scheme</i>	Given an unpartitioned whole and a unit fractional piece of it, <i>iterating</i> the fractional piece to produce a continuous <i>partitioned</i> whole and to determine the size of the unit fraction relative to the whole	If I give you this much [ <i>show a one-third piece and unpartitioned whole</i> ], what fraction of the candy bar would you have? (Also see <b>fig. 4a</b> and <b>b</b> .)
<i>Partitive fractional scheme</i>	Given an unpartitioned whole and a proper fractional piece of it, <i>partitioning</i> the piece to produce unit fractional pieces, <i>iterating</i> the unit fractional pieces to reproduce the proper fraction and the whole, <i>coordinating</i> unit fractions within the proper fraction and the whole (units coordinating at two levels) to determine the size of the proper fraction relative to the whole	If I give you this much [ <i>show an unpartitioned two-thirds piece and unpartitioned whole</i> ], what fraction of the candy bar would you have? (Also see <b>fig. 5</b> .)

a candy bar), or a discrete composite unit (e.g., a collection of twelve marbles) produces equal parts by marking separations within the unit.

One aspect of partitions that can confuse students is the idea that partitioning a unit into  $n$  parts requires only  $n - 1$  marks, leading some students to claim they have produced fifths, for instance, instead of sixths. We have also observed students counting partitions, including the marks at either end of a partitioned whole, instead of pieces, which leads them to conclude they have produced sevenths, say, instead of sixths.

- *Disembedding* occurs when a student imagines pulling out copies of some number of parts within the partitioned whole, while leaving the whole intact. So, a student might produce three-fourths by partitioning the whole into four parts and disembedding three of those parts. “Disembedding is the fundamental mental operation on which part-whole comparisons are based” (Steffe and Olive 1996, p. 118). We will discuss

part-whole schemes and other critical fraction schemes in the next section.

- *Iterating* involves repeating a part to produce identical copies of the original. These copies might be connected to form a new continuous fraction. For example, a student might imagine iterating one-fifth of a candy bar three times to produce three-fifths of that candy bar. In other cases, a student might iterate a discrete composite unit to form a new discrete collection, constituting some fraction of the original collection. In any case, students using iteration understand that iteration of the original part produces parts identical to the original in that any one of the parts can be used to substitute for any other.

### Fraction schemes

In this section, we describe several of Steffe’s fraction schemes, along with analyzed samples of student work. The samples also serve as examples of tasks teachers might pose to students to test for corresponding schemes and to better understand students’ reasoning with fractions. We intend for the analysis to serve as a guide for interpreting student responses to such tasks. We summarize the schemes, along with similar tasks, in **table 2**.

**STUDENTS’ WAYS  
OF OPERATING ARE  
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AND SENSE MAKING**

## Partitioning schemes

Partitioning is the key operation of a *simultaneous partitioning scheme*, in which a student mentally projects a composite unit into a continuous whole, using the composite unit as a “partitioning template” (Steffe 2003, p. 239) with the goal of producing equal parts. For example, a student might produce five equal parts within a whole by imagining five equally spaced units within the whole. If a student further understands that the parts are identical, so that any one of them can be iterated enough times to reproduce the whole, the student is said to have an *equi-partitioning scheme*; understanding that the resulting pieces are identical to one another is implicit, but there is no explicit goal of reestablishing the whole by iterating any of these pieces. Consider the response illustrated in **figure 2**.

The work of Student 1 indicates he used a partitioning operation to successfully resolve his task. He apparently used this operation in a manner consistent with either a simultaneous partitioning scheme (projecting a composite unit of six parts into the continuous whole) or an equi-partitioning scheme. It appears the student first made some marks on the top of the bar but was unsatisfied with them. The student then apparently erased the original marks and adjusted them accordingly. We might infer that this student iterated his initial guess to test whether it would equally partition the whole into six parts. Such a use of iterating indicates an equi-partitioning scheme, but we warn against trying to directly teach such ways of operating. Direct instruction might lead to students’ constructions of new procedures or strategies but not to meaningful ways of operating—in this case, iterating a part to check whether it can be used to create six equal parts. Students must recognize the need to iterate, and teachers can facilitate such development by, instead, asking students questions: “How do you know that is  $1/6$ ? If six people were sharing this (candy) bar, would that be a fair share? Show me that it is fair.”

Partitioning schemes provide students with a basis for developing their first fraction schemes, such as a part-whole scheme. In fact, the next two schemes we discuss can be considered reorganizations of partitioning schemes.

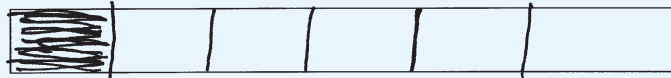
## Part-whole scheme

Students attributed with a part-whole scheme conceive of partitioned fractions as so many pieces in the partitioned fraction out of so many pieces in the partitioned whole. This scheme relies on operations of identifying (unitizing) a whole, partitioning the

**Figure 2**

### Student 1’s partitioning task

8. Shade in  $1/6$  of the bar.



**Figure 3**

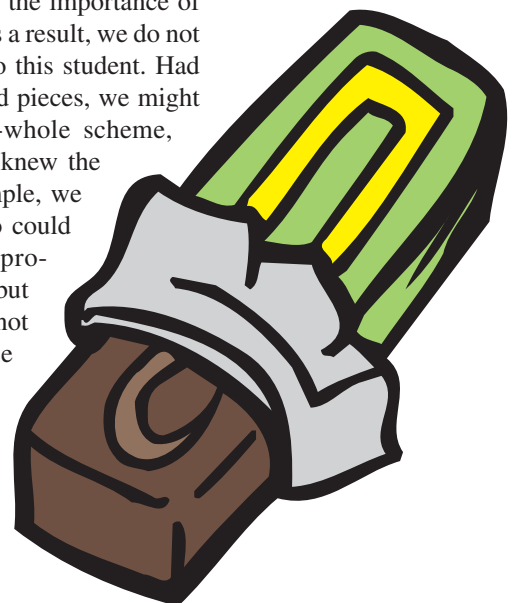
### Student 2’s part-whole task

7. Shade in  $3/5$  of the bar.



whole into equal pieces, and disembedding some number of pieces from the partitioned whole. Although the pieces in the fraction and the whole are all the same size, they may not be identical, in the sense that students may not be able to substitute one piece for another. This is because part-whole schemes are based on reorganizing a partitioning scheme, such as a simultaneous partitioning scheme, but are not based on an equi-partitioning scheme. Consider the task illustrated in **figure 3**.

Examining the work of Student 2, we infer that the student can relate fractional language to the task: He knew that the denominator indicated the number of pieces in the whole, and the numerator indicated the number of those pieces in the fraction. However, the pieces he produced are not equal, and so we wonder if he understands the importance of producing equal-sized pieces. As a result, we do not attribute a part-whole scheme to this student. Had the student produced equal-sized pieces, we might infer that he had used a part-whole scheme, but we might not infer that he knew the pieces were identical. For example, we have worked with students who could use part-whole reasoning to produce the appropriate picture; but many of these students could not identify one-fifth of the whole within that picture, implying that the students could produce equal partitions (perhaps using a simultaneous partitioning scheme), but they did not have an equi-partitioning scheme.



## Partitive unit fractional scheme

Students limited to a part-whole scheme rely on a partitioned whole and cannot identify the size of a given fractional part by iterating it within an unpartitioned whole. The transition to such treatment of the composite whole yields a *partitive unit fractional scheme* (Steffe 2003, p. 242). Most of the students we have worked with in grades 5 and 6 have developed part-whole schemes but have not developed partitive unit fractional schemes. Data from the 2003 National Assessment of Educational Progress supports a generalization of this conclusion (Kastberg and Norton 2007). We attribute this, in part, to an almost singular focus on part-whole tasks in fractions curricula.

The partitive unit fractional scheme is based on a reorganization of an equi-partitioning scheme (partitioning that produces identical parts). In utilizing a partitive unit fractional scheme, a student understands that any unit fractional part can be iterated so many times to reproduce the whole and that this number of iterations determines the *size* of the fraction relative to the whole. Furthermore, the partitive

unit fractional scheme “establishes a one-to-many relation between the part and the partitioned whole” and involves “explicit use of fractional language to refer to that relation” (Steffe 2002, p. 292). In addition to introducing fractional language, reorganizing an equi-partitioning scheme into a partitive unit fractional scheme introduces a new goal. As a partitioning scheme, the primary goal for an equi-partitioning scheme was to produce equal (and identical) parts; as a fractions scheme, the primary goal for a partitive unit fractional scheme is to determine fractional size relative to the whole, through iteration.

Both a part-whole scheme and a partitive unit fractional scheme generate fractional language, but the difference between the powers of the schemes is evident in resolving the task illustrated in **figures 4a** and **4b**. Students with only a part-whole scheme cannot determine the fraction, because the whole is unpartitioned. Students with a partitive unit fractional scheme can determine the unit fraction by iterating it to reproduce the whole and naming the fraction as the reciprocal of the number of iterations. On the other hand, a partitive unit fractional scheme cannot be used to determine the fractional size of a nonunit fraction, because the iterations will not reproduce the whole (unless, of course, the fraction in question simplifies to a unit fraction; e.g., two-sixths).

The work of Student 3 (see **fig. 4a**) indicates that he used an iterating operation in a way that fits our model of the partitive unit fractional scheme. This student drew a line to indicate where the smaller bar coincided with the whole bar, and then produced identical pieces until he reproduced the whole. The student further understood how to relate this iterating process to fractional language: He knew that because he used the piece four times to reproduce the whole, the piece was one-fourth of that whole. We can reasonably attribute a partitive unit fractional scheme to Student 3. We may wonder if his understanding generalizes to nonunit fractions.

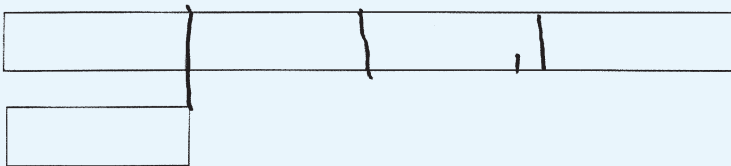
Student 4 left fewer residual markings for us to analyze (see **fig. 4b**). It seems her first step was similar to that of Student 3: She marked where the small piece coincided with the whole piece. However, her work does not clearly indicate how she went on to use this mark to arrive at the correct answer. We are left wondering whether the student used an iterating operation. One alternate hypothesis a teacher may propose is that Student 4 repeatedly partitioned the whole into various pieces (halves, thirds, fourths, etc.) until she arrived at a partitioning that coincided with the mark she made.

**Figure 4**

### Partitive unit fractional tasks

(a) Student 3’s work fits our model.

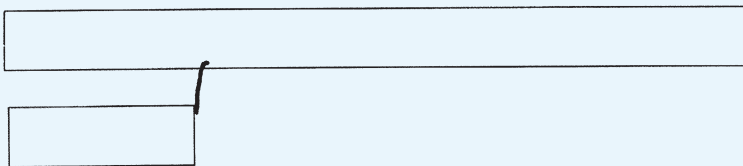
7. If the longer bar is a whole bar, what fraction is the shorter bar?



Answer:  $\frac{1}{4}$

(b) Observing Student 4’s actions would give us more insight than a paper “snapshot” does.

7. If the longer bar is a whole bar, what fraction is the shorter bar?



Answer:  $\frac{1}{4}$

This conjecture may seem far-fetched, because few teachers would solve the problem using this tedious approach, but we have witnessed students using this more basic partitioning strategy to do just that.

Student 4 did not draw the marks we would expect to support the hypothesis described above. It seems more reasonable that she performed some sort of iterating operation with a physical motion instead of her pencil, such as by moving her finger along the whole bar, repeating the size of that first piece, until she realized that four of those pieces would constitute the whole. Because this conjecture seems more likely, given the single mark she left, we conclude that she used a partitive unit fractional scheme. This is an example in which her teacher would probably want to give this student a similar task and observe the student solving it. Seeing the student's actions would yield insight that we lack from this paper "snapshot."

### Partitive fractional scheme

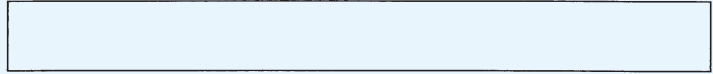
The *partitive fractional scheme* is a generalization of the partitive unit fractional scheme. Students can use the more general scheme to conceive of a proper fraction, such as three-fifths, as three of one-fifth of the whole. This involves producing composite fractions from unit fractions through iteration, while maintaining the relation between the unit fraction and the whole. It also involves *units coordination* at two levels, because the student must coordinate three-fifths as three iterations of the fractional unit and the whole as five iterations of the fractional unit. In other words, three-fifths is a unit of three fractional units, and the whole is a unit of five fractional units. Consider the coordination indicated by the student response in **figure 5**.

If we were only interested in correctness, then we could quickly assess the response of Student 5 as incorrect and move on. However, we are more interested in how students reason. Her answer is not far from the correct one, and the few marks she made warrant further consideration. It is not clear how she decided that the shorter bar should be partitioned into three pieces, but we can see that these three pieces are reasonably equivalent in size. More interestingly, Student 5 seemed to know that three pieces in the shorter bar are actually fourths of the whole bar. She was reasonably successful in simultaneously dealing with thirds and fourths, concluding that the shorter piece is three-fourths of the whole bar. This indicates she was coordinating units of units. Although three-fourths is not correct, the work of Student 5 indicates she was operating

**Figure 5**

#### Student 5's partitive fractional task

9. If the longer bar is a whole bar, what fraction is the shorter bar?



Answer:  $\frac{3}{4}$

in a way consistent with our understanding of the partitive fractional scheme.

### Using Schemes

We have introduced some of Steffe's fraction schemes in hopes that teachers will find them useful in assessing and understanding their students' ways of operating with fractions. But we have warned against trying to teach these schemes directly. Students' ways of operating are deeply connected to their meaning and sense making, and attempts to circumvent or replace those ways of operating can only yield disconnected knowledge. Rather, teachers should find ways to work within their students' operating methods while generating a need to develop more powerful ways of operating. For example, tasks such as the one illustrated in **figure 4** can be presented with manipulatives, such as Cuisenaire rods.

Students with partitive unit fractional schemes should be able to estimate the size of the smaller rod relative to the longer (whole) rod. This ability to estimate might generalize to nonunit fractions and contribute to proportional reasoning (Nabors 2003). Students limited to a part-whole understanding of fractions might also engage in such tasks by using the smaller rod to *segment* the larger rod (i.e., place copies of the smaller rod side by side until they equal the length of the larger rod). With

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the larger rod partitioned by the smaller rods, these students can use their part-whole schemes to determine the fraction. Segmenting can be a pathway to developing the iterative operation of an equipartitioning scheme and a partitive unit fractional scheme. The key is for students to have opportunities to carry out such experimental actions, generating new ways of operating from their existing ways of operating.

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## References

- Kastberg, Signe, and Anderson H. Norton. "Building a System of Rational Numbers." In *Results and Interpretations of the 2003 Mathematics Assessment of the National Assessment of Educational Progress*, edited by Peter Kloosterman and Frank K. Lester Jr., 67–94. Reston: National Council of Teachers of Mathematics, 2007.
- Nabors, Wanda K. "From Fractions to Proportional Reasoning: A Cognitive Schemes of Operation Approach." *Journal of Mathematical Behavior* 22 (2003): 133–79.
- Olive, John. "From Fractions to Rational Numbers of Arithmetic: A Reorganization Hypothesis." *Mathematical Thinking and Learning* 1, no. 4 (1999): 279–314.
- Steffe, Leslie P. "A New Hypothesis Concerning Children's Fractional Knowledge." *Journal of Mathematical Behavior* 20 (2002): 267–307.
- . "Fractional Commensurate, Composition, and Adding Schemes Learning Trajectories of Jason and Laura: Grade 5." *Journal of Mathematical Behavior* 22 (2003): 237–95.
- Steffe, Leslie P., Paul Cobb, and Ernst von Glasersfeld. *Construction of Arithmetical Meanings and Strategies*. New York: Springer-Verlag, 1988.
- Steffe, Leslie P., and John Olive. "Symbolizing As a Constructive Activity in a Computer Microworld." *Journal of Educational Computing Research* 14 (1996): 103–28.
- U.S. Math Recovery Council (USMRC). 2008. [www.mathrecovery.org](http://www.mathrecovery.org).
- Wright, Robert J. "Professional Development in Recovery Education." In *Radical Constructivism in Action*, edited by Leslie P. Steffe and P. W. Thompson, 134–49. London: Routledge Falmer, 2000.
- . "Mathematics Recovery: A Program of Intervention in Early Number Learning." *Australian Journal of Learning Disabilities* 8, no. 4 (2003):6–11. ▲